

# Essential surfaces in the exteriors of torus knots with twists on 2-strands

by

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**Abstract.** In the present paper, we will show that torus knots with twists on 2-strands contain no closed essential surfaces in the exteriors. As an application, we will show that Hoiden-Morimoto's inequality for 1-bridge genus of knots is best possible.

Key words and phrases : torus knots, small knots, meridionally small knots.

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## 1. Introduction

Let  $K$  be a knot in the 3-sphere  $S^3$ ,  $N(K)$  the regular neighborhood of  $K$  in  $S^3$  and  $E(K) = cl(S^3 - N(K))$  the exterior. Let  $F$  be a surface (i.e. a connected 2-manifold) properly embedded in  $E(K)$ . Then we say that  $F$  is a closed essential surface if  $F$  is closed, incompressible and not parallel to the torus  $\partial E(K)$ , and that  $F$  is a meridionally essential surface if  $\partial F \neq \emptyset$ , each component of  $\partial F$  is a meridian of  $N(K)$ ,  $F$  is incompressible and not parallel to an annulus in  $\partial E(K)$ . We say that a knot  $K$  is small if  $E(K)$  contains no closed essential surfaces, and that a knot  $K$  is meridionally small if  $E(K)$  contains no meridionally essential surfaces. We note that if a knot  $K$  in  $S^3$  is small then it is meridionally small by [CGLS, Theorem 2.0.3].

For coprime integers  $p, q$ , we denote the torus knot of type  $(p, q)$  by  $T(p, q)$ . Let  $r$  be an arbitrary integer, then we consider the knot obtained from  $T(p, q)$  by adding  $r$ -times full twists at mutually parallel 2-strands to  $T(p, q)$  as illustrated in Figure 1, and denote it by  $K(p, q; r)$  (c.f. [MSY]). For the definition of torus knots we refer [Ro]. In the present paper, we will show :

**Theorem 1.1** *For any coprime integers  $p, q$ , and for any integer  $r$ ,  $K(p, q; r)$  is small and meridionally small.*

**Remark 1** Since small knots are meridionally small as we mentioned above, it is sufficient to show that  $K(p, q; r)$  is small. However, by some technical reason, we will show the smallness and the meridional smallness simultaneously.

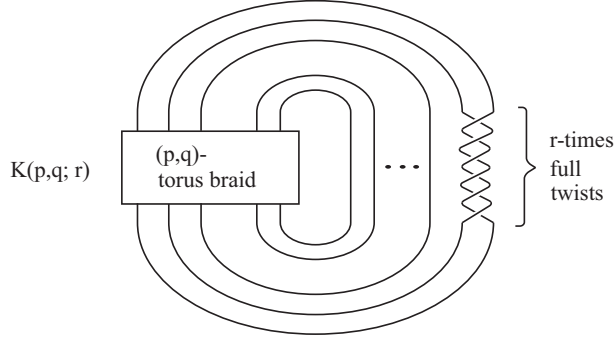


Figure 1

**Remark 2** In the present paper, we deal with torus knots with twists on 2-strands. In general, we can deal with torus knots with twists on arbitrary strands. In the paper [MY], we deal with such knots, and show that, for any composite number  $n$ , there are infinitely many torus knots with twists on  $n$ -strands containing essential tori in the exteriors([MY, Theorem 2.2]).

We say that  $(V_1, V_2)$  is a Heegaard splitting of  $S^3$  if  $S^3 = V_1 \cup V_2$ ,  $V_1 \cap V_2 = \partial V_1 = \partial V_2$  and both  $V_1$  and  $V_2$  are handlebodies. The genus of  $V_1$  (= the genus of  $V_2$ ) is called the genus of the Heegaard splitting and the surface  $\partial V_1 = \partial V_2$  is called the Heegaard surface of the Heegaard splitting. Then for any knot  $K$  in  $S^3$  it is well known that there is a Heegaard splitting  $(V_1, V_2)$  of  $S^3$  such that  $K$  intersects  $V_i$  in a single trivial arc in  $V_i$  for both  $i = 1, 2$ . Hence we define the 1-bridge genus  $g_1(K)$  of  $K$  as the minimal genus among all such Heegaard splittings  $(V_1, V_2)$  of  $S^3$ . For two knots  $K_1, K_2$  in  $S^3$ , we denote the connected sum of  $K_1$  and  $K_2$  by  $K_1 \# K_2$ . Then by a little observation, we immediately see the following :

**Fact 1.2** For any two knots  $K_1$  and  $K_2$  in  $S^3$ ,  $g_1(K_1 \# K_2) \leq g_1(K_1) + g_1(K_2)$ .

On the problem to estimate the lower bound of  $g_1(K_1 \# K_2)$ , Hoiden and the author showed :

**Theorem 1.3** ([Ho, Theorem], [M3, Theorem 3.10]) Let  $K_1, K_2$  be two knots in  $S^3$ . If  $K_1$  and  $K_2$  are meridionally small, then  $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1$ .

As an application of Theorem 1.1, we will show that the above inequality is best possible as follows (note that smallness implies meridionally smallness) :

**Theorem 1.4** There are infinitely many pairs of small knots  $K_1, K_2$  in  $S^3$  with  $g_1(K_1 \# K_2) = g_1(K_1) + g_1(K_2) - 1$ .

**Proof** Let  $t(K)$  be the tunnel number of a knot  $K$ , i.e.,  $t(K)$  is the minimum number of arcs properly embedded in  $E(K)$  such that the exterior of the arcs is homeomorphic to a handlebody. Then it is well known that the inequality  $t(K) \leq g_1(K) \leq t(K) + 1$  holds for any knot  $K$ . To show Theorem 1.4, we need the following lemma.

**Lemma 1.5** ([M2, Proposition 1.7]) *Let  $K$  be a knot with  $g_1(K) = t(K) + 1$ . Then  $g_1(K \# K') \leq g_1(K)$  holds for any 2-bridge knot  $K'$ .*

Now, let  $m$  be an integer and consider the knot  $K_1 = K(7, 17, 5m - 2)$ , i.e., consider the case when  $p = 7, q = 17$  and  $r = 5m - 2$ . Then, by Theorem 1.1,  $K_1$  is small. Moreover, by [MSY, Theorem 2.1],  $t(K_1) = 1$  and  $g_1(K_1) = 2$ , i.e.,  $g_1(K_1) = t(K_1) + 1$ . Let  $K_2$  be a (non-trivial) 2-bridge knot in  $S^3$ . Then it is well known that  $K_2$  is small and  $g_1(K_2) = 1$ . Then by Lemma 1.5,  $g_1(K_1 \# K_2) \leq g_1(K_1) = 2$ . On the other hand,  $g_1(K_1 \# K_2) \geq 2$  because 1-bridge genus one knots are tunnel number one knots, and hence prime by [No, Sc]. Thus  $g_1(K_1 \# K_2) = 2$  and  $g_1(K_1 \# K_2) = g_1(K_1) + g_1(K_2) - 1$  for the small knots  $K_1$  and  $K_2$ . This completes the proof of Theorem 1.4.  $\square$

By the way, for the additivity of tunnel number, it is well known that  $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ . Concerning the best possibility of this inequality, by putting  $K_m = K(7, 17, 5m - 2)$  and  $K_n = K(7, 17, 5n - 2)$ , we have shown in [MSY] that  $t(K_m) = t(K_n) = 1$  and  $t(K_m \# K_n) = 3$ , i.e.,  $t(K_m \# K_n) = t(K_m) + t(K_n) + 1$ . Moreover, related to this problem, we have shown the following theorem in [M1].

**Theorem 1.6** ([M1, Theorem 1.6]) *Let  $K_1$  and  $K_2$  be both meridionally small knots. Then  $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$  if and only if  $g_1(K_i) = t(K_i) + 1$  for both  $i = 1, 2$ .*

By Theorem 1.1, we see that both  $K_m$  and  $K_n$  are meridionally small, and this shows that there are infinitely many pairs of knots satisfying the hypothesis and the conclusion of the above theorem.

Throughout the present paper, we will work in the piecewise linear category. For a manifold  $X$  and a subcomplex  $Y$  of  $X$ , we denote a regular neighborhood of  $Y$  in  $X$  by  $N(Y, X)$  or simply  $N(Y)$ .

## 2. Preliminaries for the proof of Theorem 1.1

Let  $V_1$  be the standard solid torus in  $S^3$ , and take a torus knot  $T(p, q)$  in  $\partial V_1$  so that it winds around  $V_1$  with  $p$ -times in the longitudinal direction and  $q$ -times in the meridional direction. Let  $D_1$  be a 2-disk, and put  $V_2 = D_1 \times I$  with  $D_1 = D_1 \times \{\frac{1}{2}\}$ ,  $D_1^0 = D_1 \times \{0\}$  and  $D_1^1 = D_1 \times \{1\}$ , where  $I = [0, 1]$  is the unit interval. Remove parallel two subarcs from  $T(p, q)$  and attach the 1-handle  $V_2$  to  $V_1$  so that the two

subarcs removed can be regarded as two arcs in  $\partial V_2$ . Then we can regard  $T(p, q)$  as a knot in the genus two surface  $\partial(V_1 \cup V_2)$  such that the knot intersects  $D_1$  in exactly two points. Perform  $r$ -times full twists at  $D_1$ , then we get the knot  $K(p, q; r)$  in  $\partial(V_1 \cup V_2)$  as in Figure 2.

Put  $K = K(p, q; r)$  and we may assume  $p > 0$ . If  $r = 0$ , then  $K$  is the torus knot  $T(p, q)$  and is small (and meridionally small). Hence hereafter, we assume  $r \neq 0$ . Moreover, if  $r < 0$ , then by taking  $K(p, -q; -r)$  instead of  $K(p, q; r)$  we may assume that  $r > 0$  because  $E(K(p, -q; -r))$  is homeomorphic to  $E(K(p, q; r))$ . Let  $D_2$  be a 2-disk properly embedded in  $cl(S^3 - (V_1 \cup V_2))$  such that  $D_1 \cap D_2 =$  a point and  $D_2 \cap (K - V_2) = \emptyset$  as in Figure 2. Put  $\partial D_2 = \gamma_1 \cup \gamma_2$  such that  $\gamma_1 = \partial D_2 \cap V_1$  and  $\gamma_2 = \partial D_2 \cap V_2$ . By these constructions and definitions, we see that  $K \cap D_1 = K \cap \partial D_1 = 2r$  points as in Figures 2 and 3.

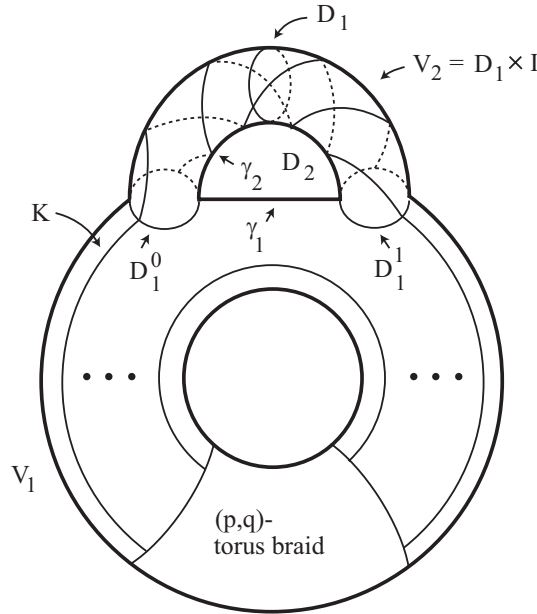


Figure 2

Suppose  $K$  is not small or not meridionally small, then there is a closed essential surface or a meridionally essential surface properly embedded in  $E(K)$ . If the surface is closed, then denote it by  $F$ . If the surface is not closed, then by capping the boundary components with meridian disks of  $N(K)$ , we get a closed surface and denote it by  $F$ . Hence we see that if  $F \cap K = \emptyset$  then  $F - K = F$  is incompressible in  $S^3 - K$  and not a torus parallel to  $\partial E(K)$ , and that if  $F \cap K \neq \emptyset$  then  $F - K$  is incompressible in  $S^3 - K$  and  $F$  is not a 2-sphere bounding a 3-ball intersecting  $K$  in a trivial arc.

**Lemma 2.1** *Suppose there is a disk  $D$  in  $S^3$  such that  $D \cap K$  is a single point*

and  $D \cap F = \partial D$  is essential in  $F$  or bounds such a disk  $D'$  in  $F$  that the 2-sphere  $D \cup D'$  does not bound a 3-ball intersecting  $K$  in a trivial arc. Let  $F'$  be a closed surface(s) obtained from  $F$  by a compression along  $D$ . Then  $F' \cap K \neq \emptyset$ ,  $F' - K$  is incompressible in  $S^3 - K$ , and each component of  $F'$  is not a 2-sphere bounding a 3-ball intersecting  $K$  in a trivial arc.

**Proof.** If  $F' - K$  is compressible in  $S^3 - K$ , then there is a compressing disk of  $F' - K$ , say  $D'$ , such that  $D' \cap K = \emptyset$  and  $D' \cap F' = \partial D'$  is an essential loop in  $F' - K$ . Then, since  $F$  is obtained from  $F'$  by tubing along a subarc of  $K$ ,  $D'$  is a compressing disk of  $F - K$  in  $S^3 - K$ , a contradiction. If  $F'$  is a 2-sphere bounding a 3-ball intersecting  $K$  in a trivial arc, then  $F$  is a torus parallel to  $\partial E(K)$ , a contradiction. If  $F'$  consists of two components and at least one component is a 2-sphere bounding a 3-ball intersecting  $K$  in a trivial arc, then this means that  $\partial D$  bounds such a disk  $D'$  in  $F$  that the 2-sphere  $D \cup D'$  bounds a 3-ball intersecting  $K$  in a trivial arc, a contradiction. This completes the proof of Lemma 2.1.  $\square$

Now, perform the compression as many as possible if  $F$  has disks as in Lemma 2.1, and we denote ( a component of ) the resulting closed surface(s) by  $F$  again. Then, by Lemma 2.1,  $F$  is a closed surface in  $S^3$  such that  $cl(F - N(K))$  is essential in  $E(K)$ , and hence we say that  $F - K$  is essential in  $S^3 - K$ .

By the incompressibility of  $F - K$  in  $S^3 - K$ , we may assume that  $D_1 \cap F$  consists of  $n$  arcs for some  $n \geq 0$  and  $V_2 \cap F$  consists of  $n$  rectangles, where each arc of  $D_1 \cap F$  separates the two points  $D_1 \cap K = \partial D_1 \cap K$ . We assume that  $n$  is minimal among all such surfaces  $F$ . Put  $V_3 = cl(S^3 - (V_1 \cup V_2))$ . Then  $V_3$  is a genus two handlebody and  $(V_1 \cup V_2, V_3)$  is a genus two Heegaard splitting of  $S^3$ . Put  $V_1 \cap F = F_1$ ,  $V_2 \cap F = F_2$  and  $V_3 \cap F = F_3$ . Put  $\partial\gamma_1 = \partial\gamma_2 = \{p_1, p_2\}$ , and let  $d_i (i = 1, 2)$  be the subarc of  $\gamma_2$  such that  $\partial d_i = \{p_i, q_i\}$  with  $d_i \cap K = q_i$  as illustrated in Figure 3.

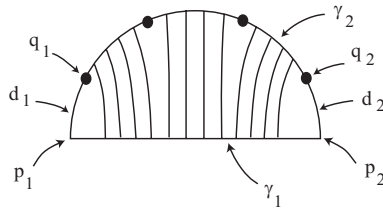


Figure 3

**Lemma 2.2** We may assume that each component of  $F_3 \cap D_2$  is an arc connecting  $\gamma_2$  and  $\gamma_1$ , and is disjoint from  $d_1 \cup d_2$  as in Figure 3.

**Proof.** By the incompressibility of  $F - K$  in  $S^3 - K$ , we may assume that each

component of  $F_3 \cap D_2$  is an arc. If there is a component of  $F_3 \cap D_2$  such that at least one end point is in  $d_1$  or in  $d_2$ , then we can slide the end point along the arc  $d_1$  or  $d_2$  through  $p_1$  or  $p_2$  until the interior of  $\gamma_1$ . This sliding can be realized by an ambient isotopy of  $F - K$  in  $S^3 - K$ . Hence we may assume that  $d_1 \cup d_2$  has no point of  $F_3 \cap \partial D_2$ . At this stage, by performing a suitable deformation of  $F - K$  after the above sliding if necessary, we may assume that  $F_2 = V_2 \cap F$  consists of rectangles.

Let  $\alpha$  be an outermost arc component of  $F_3 \cap D_2$  in  $D_2$ ,  $\Delta$  the outermost disk and put  $\beta = cl(\partial\Delta - \alpha)$ . If  $\partial\alpha \subset \gamma_1$ , then  $\beta \subset \gamma_1$  and we can eliminate the arc  $\alpha$  by sliding along  $\Delta$ . Suppose  $\partial\alpha \subset \gamma_2$ , then  $\beta \subset \gamma_2$ . First suppose  $\beta \cap K = \emptyset$ . Then  $\beta$  connects two adjacent rectangles of  $F_2$ , say  $R_1$  and  $R_2$ . Perform a boundary compression of  $F_3$  along  $\Delta$ , and let  $b$  be the band in  $V_2$  produced by the boundary compression. Then, since  $R_1 \cup b \cup R_2$  is a  $\partial$ -parallel disk in  $V_2$  and the parallelism is disjoint from  $K$ , we can push  $R_1 \cup b \cup R_2$  out from  $V_2$ . This decrease the number  $n$ , a contradiction.

Next suppose  $\beta \cap K \neq \emptyset$ . Then  $\beta \cap K$  consists of a single point of  $K \cap \gamma_2$ , because each component of  $F_2$  separates the two components of  $K \cap V_2$ . Then  $\beta$  meets a single rectangle of  $F_2$ , say  $R$ , and let  $\Delta'$  be the boundary compressing disk of  $R$  in  $V_2$  with  $\Delta' \cap \partial V_2 = \beta$ . Then  $D = \Delta \cup \Delta'$  is a disk such that  $D \cap F = \partial D$  and  $D$  intersects  $K$  in a single point. Since  $F$  has no disk as in Lemma 2.1,  $\partial D$  bounds a disk in  $F$ , say  $D'$ , such that  $D \cup D'$  bounds a 3-ball intersecting  $K$  in a trivial arc. Hence by sliding  $F$  along the 3-ball, we can eliminate the rectangle  $R$ . This contradicts the minimality of  $n$ . After all, we see that each component  $\alpha$  of  $F_3 \cap D_2$  is an arc connecting  $\gamma_2$  and  $\gamma_1$  with  $\alpha \cap (d_1 \cup d_2) = \emptyset$ . This completes the proof of Lemma 2.2.  $\square$

Let  $K \cap V_1 = K \cap \partial V_1 = k_1 \cup k_2$  be two subarcs of  $K$  and put  $\partial k_1 = \{x_1, y_1\}$ ,  $\partial k_2 = \{x_2, y_2\}$  so that  $D_1^0 \cap (k_1 \cup k_2) = \{x_1, x_2\}$  and  $D_1^1 \cap (k_1 \cup k_2) = \{y_1, y_2\}$ . A schematic picture of  $\{k_1, k_2, D_1^0, D_1^1, \gamma_1\}$  in  $\partial V_1$  is illustrated in Figure 4.

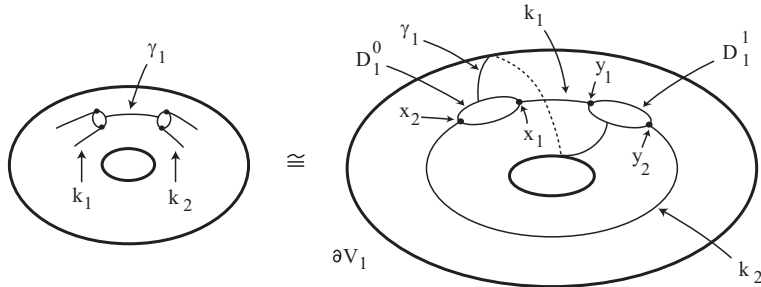


Figure 4

**Lemma 2.3** *We may assume the following :*

- (1) *There is no pair of a subarc  $\alpha$  of  $k_1 \cup k_2$  and an arc  $\beta$  properly embedded in  $F_1$*

such that  $\alpha \cap \beta = \partial\alpha = \partial\beta$  and  $\alpha \cup \beta$  bounds a disk in  $V_1$ .

(2) There is no 2-gon in  $\partial F_1 \cup (k_1 \cup k_2)$  which bounds a disk in  $\partial V_1$ .

(3) There is no 2-gon in  $\partial F_1 \cup \gamma_1$  which bounds a disk in  $\partial V_1$ .

(4) There is no 2-gon in  $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$  which bounds a disk in  $\partial V_1$ .

**Proof.** (1) Suppose there is such a pair  $(\alpha, \beta)$ , and let  $\Delta$  be the disk in  $V_1$  with  $\partial\Delta = \alpha \cup \beta$ . Let  $N(\Delta)$  be a regular neighborhood of  $\Delta$  in  $S^3$  such that  $N(\Delta) \cap F$  is a disk which is a regular neighborhood of  $\beta$  in  $F$ , say  $N(\beta, F)$ . Put  $c = \partial N(\beta, F)$ , then since  $c$  is a loop in  $\partial N(\Delta)$  and  $N(\beta, F) \cap K =$  two points,  $c$  bounds a disk in  $\partial N(\Delta)$  disjoint from  $K$ . By the incompressibility of  $F - K$  in  $S^3 - K$ ,  $c$  bounds a disk, say  $B$ , in  $F - K$ . Then  $F = B \cup N(\beta, F)$  is a 2-sphere bounding a 3-ball intersecting  $K$  in a trivial arc, a contradiction.

(2) If there is such a 2-gon in  $\partial F_1 \cup (k_1 \cup k_2)$ , then we can find a pair  $(\alpha, \beta)$  satisfying the condition (1), a contradiction.

(3) Suppose there is a 2-gon  $\alpha \cup \beta$  in  $\partial F_1 \cup \gamma_1$  with  $\alpha \subset \partial F_1$  and  $\beta \subset \gamma_1$ . Then by the ambient isotopy along the disk bounded by  $\alpha \cup \beta$ ,  $F$  is deformed so that  $F_3 \cap D_2$  contains an arc whose end points are in  $\gamma_2$ . Then by the argument in the proof of Lemma 2.2, we have a contradiction.

(4) Suppose there is a 2-gon  $\alpha \cup \beta$  such that  $\alpha \subset \partial F_1$  and  $\beta \subset ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$ . If  $\alpha \subset D_1^0$  (or  $D_1^1$ ), then we have a contradiction because each component of  $\partial F_1 \cap D_1^0$  (or  $\partial F_1 \cap D_1^1$  resp.) is an arc which separates  $x_1$  and  $x_2$  (or  $y_1$  and  $y_2$  resp.). Suppose  $\alpha \cap (D_1^0 \cup D_1^1) = \partial\alpha$ , and let  $\Delta$  be the disk in  $\partial V_1$  with  $\partial\Delta = \alpha \cup \beta$ . Then by the ambient isotopy of  $F$  along  $\Delta$ , we get a band in  $V_2$  connecting two rectangles in  $F_2$ . Then by the argument in the proof of Lemma 2.2, we have a contradiction. This completes the proof of Lemma 2.3.  $\square$

By the incompressibility of  $F_1$  in  $V_1$ , each component of  $F_1$  is a  $\partial$ -parallel disk, a  $\partial$ -parallel annulus or a meridian disk of  $V_1$ .

**Lemma 2.4** *Let  $E$  be a  $\partial$ -parallel disk component of  $F_1$ , and  $E'$  the disk in  $\partial V_1$  to which  $E$  is parallel. Then we may assume that one of the following holds :*

(1)  $E'$  is a small regular neighborhood of  $k_i$  in  $\partial V_1$  ( $i = 1, 2$ ) (Figure 5(1)),

(2)  $E'$  is a small regular neighborhood of  $D_1^0 \cup k_i \cup D_1^1$  in  $\partial V_1$  ( $i = 1, 2$ ) (Figure 5(2)).

**Proof.** Put  $X = \{x_1, x_2, y_1, y_2\}$ , and we divide the proof into the following five cases.

Case (0) :  $E' \cap X = \emptyset$ . In this case, if  $E' \cap (D_1^0 \cup D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) \neq \emptyset$ , then there is a 2-gon in  $\partial E' \cap ((\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) - X)$ . This contradicts Lemma 2.3. Thus  $E' \cap (D_1^0 \cup D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) = \emptyset$  and we can eliminate the disk  $E$ .

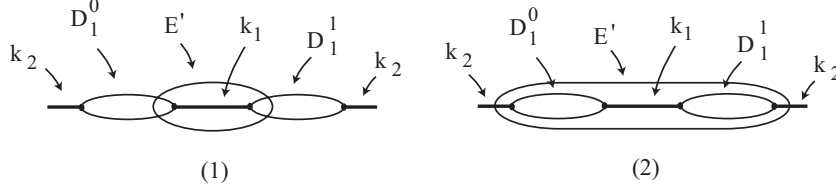


Figure 5

Case (1) :  $E' \cap X = \text{one point}$ . In this case, by Lemma 2.3,  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is a wheel with three edges as in Figure 6(1). Then by the deformation along the arrow indicated in Figure 6(2), we can decrease the number  $n$ , a contradiction.

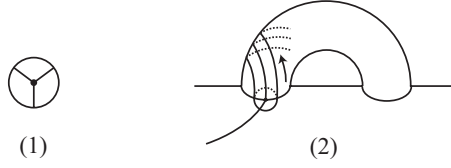


Figure 6

Case (2) :  $E' \cap X = \text{two points}$ . In this case, by Lemma 2.3,  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is one of the three patterns illustrated in Figure 7, i.e., (i) the two points are separated, (ii) the two points are connected by an arc, (iii) the two points are connected by two arcs. In addition, by Case (1), we may assume that  $E'$  contains no other disks.

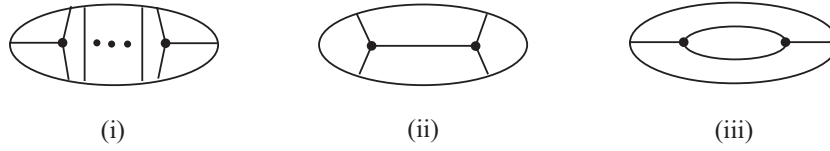


Figure 7

In case (i), by a boundary compression of  $E$ , we get a  $\partial$ -parallel disk satisfying the condition in Case (1). Then we get a contradiction as in Case (1). In case (ii), if the arc connecting the two points is a subarc of  $\partial(D_1^0 \cup D_1^1)$ , then this is a contradiction because each arc of  $D_1^0 \cap F_1$  (or  $D_1^1 \cap F_1$ ) separates the two points  $x_1$  and  $x_2$  (or  $y_1$  and  $y_2$  resp.). Hence the arc connecting the two points is  $k_1$  or  $k_2$ , and we get the conclusion (1). In case (iii), the circle made by the two arcs is  $\partial D_1^0$  or  $\partial D_1^1$ . Then we have  $n = 0$  because  $\partial D_1^0 \cap F_1 = \emptyset$  or  $\partial D_1^1 \cap F_1 = \emptyset$ . On the other hand, we have  $n > 0$  because  $\gamma_1$  connects  $\partial D_1^0$  and  $\partial D_1^1$  as in Figure 4 and  $\gamma_1$  intersects  $\partial E' = \partial E$ . This is a contradiction.

Case (3)  $E' \cap X = \text{three points}$ . If  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is not connected,

then by the argument in case (i) of Case (2), we have a contradiction. Hence  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is connected and one of the two patterns illustrated in Figure 8 happens. In addition, we may assume that  $\{x_1, x_2\}$  is contained in  $E'$ .

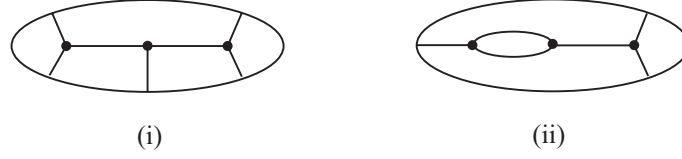


Figure 8

In case (i), one of the two arcs of  $\partial D_1^0 - \{x_1, x_2\}$  is contained in  $E'$ . Then the other arc and  $\partial F_1$  makes a 2-gon in  $\partial F_1 \cup \partial D_1^0$ . This contradicts Lemma 2.3. In case (ii),  $\partial D_1^0$  is contained in  $E'$ . Since each component of  $F_1$  meeting  $D_1^0$  is a disk satisfying the conclusion (1) of this lemma, there are  $n$  such disk components because of  $|D_1^0 \cap F_1| = n$ . On the other hand,  $E$  meets  $D_1^1$  and this shows that  $|D_1^1 \cap F_1| > n$ . This contradicts that  $|D_1^0 \cap F_1| = |D_1^1 \cap F_1| = n$ .

Case (4)  $E' \cap X =$  four points. If  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is not connected, then by the arguments in Cases (1), (2), (3), we can regard  $E'$  as  $N(k_1) \cup N(k_2) \cup b$ , where  $N(k_i)$  is a small regular neighborhood of  $k_i$  in  $\partial V_1$  ( $i = 1, 2$ ) and  $b$  is a band with  $b \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2) = \emptyset$ . Then by performing a boundary compression of  $F$  along  $b$ ,  $E$  is changed to two disk components of  $F_1$  both of which satisfy the conclusion (1) of this lemma. Suppose  $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$  is connected. Then we have one of the five patterns illustrated in Figure 9.

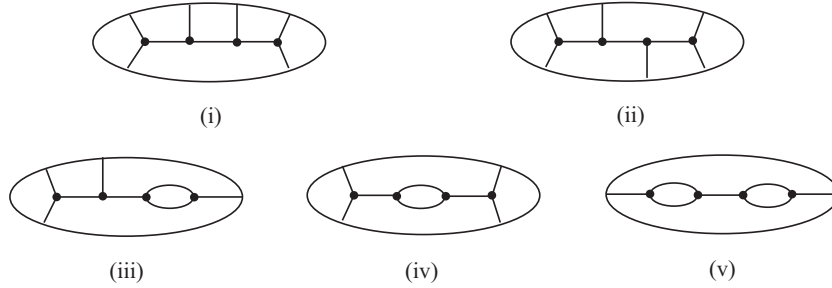


Figure 9

In cases (i), (ii), we have a contradiction as in case (i) of Case (3). In cases (iii), (iv), we have a contradiction as in case (ii) of Case (3). Hence we are in case (v) and  $E$  is a disk satisfying the conclusion (2) of this lemma. This completes the proof of Lemma 2.4.  $\square$

Put  $cl(K - (k_1 \cup k_2)) = k'_1 \cup k'_2$ , i.e.,  $k'_1$  and  $k'_2$  are the two arcs  $K \cap V_2 = K \cap \partial V_2$

such that  $k'_1$  connects  $x_1$  and  $y_2$  and  $k'_2$  connects  $x_2$  and  $y_1$ . Hereafter, by changing the letters if necessary, we assume that  $q_1$  is contained in  $k'_2$  and  $q_2$  is contained in  $k'_1$ . Moreover, we put  $\partial D_1^0 = a_1 \cup b_1 \cup c_1$  with  $\partial a_1 = \{x_1, p_1\}$ ,  $\partial b_1 = \{p_1, x_2\}$ ,  $\partial c_1 = \{x_1, x_2\}$  and  $\partial D_1^1 = a_2 \cup b_2 \cup c_2$  with  $\partial a_2 = \{y_1, p_2\}$ ,  $\partial b_2 = \{p_2, y_2\}$ , and  $\partial c_2 = \{y_1, y_2\}$  as illustrated in Figure 10.

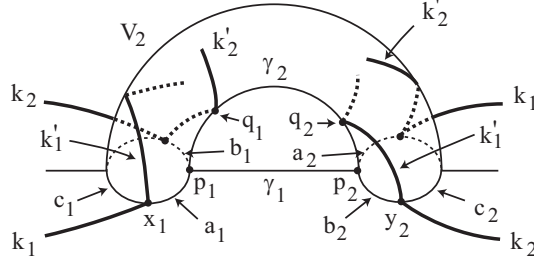


Figure 10

- Lemma 2.5** (1) *Any component of  $F_1 \cap D_1^0$  (or  $F_1 \cap D_1^1$ ) is an arc connecting a point in  $a_1$  (or  $a_2$  resp.) and a point in  $c_1$  (or  $c_2$  resp.).*  
(2) *There is no 3-gon which consists of a subarc of  $a_1$  (or  $a_2$ ), a subarc of  $\partial F_1$  and a subarc of  $\gamma_1$  such that it bounds a disk in  $\partial V_1$ .*  
(3) *There is no 3-gon which consists of a subarc of  $\partial D_1^0$  (or  $\partial D_1^1$ ), a subarc of  $\partial F_1$  and a subarc of  $k_1 \cup k_2$  such that it bounds a disk in  $\partial V_1$ .*

**Proof.** (1) Since each component of  $F_2 = F \cap V_2$  is a rectangle separating  $k'_1$  and  $k'_2$ , any component of  $F_1 \cap D_1^0$  is an arc connecting a point in  $a_1 \cup b_1$  and a point in  $c_1$ . Suppose there is an arc component of  $F_1 \cap D_1^0$  connecting a point in  $b_1$  and a point in  $c_1$ . This means that there is a subarc of  $\partial F_2$  which connects a point in  $b_1$  and a point in  $d_1$ . But  $d_1$  does not meet  $F$  by Lemma 2.2. This contradiction shows that any component of  $F_1 \cap D_1^0$  is an arc connecting a point in  $a_1$  and a point in  $c_1$ . Similarly, we see that any component of  $F_1 \cap D_1^1$  is an arc connecting a point in  $a_2$  and a point in  $c_2$ .

(2) Suppose there is such a 3-gon, and let  $\Delta$  be the disk in  $\partial V_1$  bounded by the 3-gon. Then by an ambient isotopy along  $\Delta$ , we can deform  $F$  so that  $F \cap D_2$  has an arc component whose end points are in  $\gamma_2$  as illustrated in Figure 11. Then by the argument in the proof of Lemma 2.2, we decrease the number  $n$ , a contradiction.

(3) Suppose there is such a 3-gon, and let  $\Delta$  be the disk in  $\partial V_1$  bounded by the 3-gon. Then by an ambient isotopy along  $\Delta$ , i.e., a boundary compression for  $F_1$  and a sliding along the arrow indicated in Figure 12, we can decrease the number  $n$ , a contradiction. This completes the proof of Lemma 2.5.  $\square$

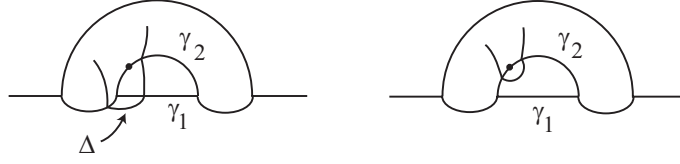


Figure 11

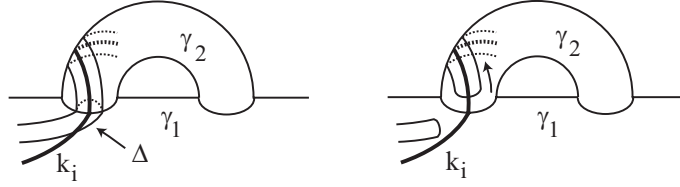


Figure 12

**Lemma 2.6** *Let  $E$  be a  $\partial$ -parallel disk component of  $F_1$ , and  $E'$  the disk in  $\partial V_1$  to which  $E$  is parallel. Then  $E'$  is a small regular neighborhood of  $k_1$  in  $\partial V_1$ . This means that in the four conclusions in Lemma 2.4 ((1), (2), (i = 1, 2)), the conclusion (1)(i = 1) only occurs.*

**Proof.** Suppose we are in the conclusion (1) of Lemma 2.4. If  $E'$  is a regular neighborhood of  $k_2$ , then there is an arc component of  $D_1^0 \cap F_2$  connecting a point in  $b_1$  and a point in  $c_1$  or connecting a point in  $a_1$  and a point in  $c_1$  with a 3-gon as in Lemma 2.5(2). This contradicts Lemma 2.5, and hence  $E'$  is a regular neighborhood of  $k_1$ .

Next suppose we are in the conclusion (2) of Lemma 2.4, and let  $E'$  be a regular neighborhood of  $D_1^1 \cup k_i \cup D_1^1$  ( $i = 1, 2$ ). Since  $\partial E \cap \gamma_1 \neq \emptyset$ , we see that  $F \cap D_2 \neq \emptyset$  and  $n > 0$ . If  $E'$  is a regular neighborhood of  $D_1^0 \cup k_2 \cup D_1^1$ , then there is a  $\partial$ -parallel disk component of  $F_1$  which is parallel to a regular neighborhood of  $k_2$  because  $n > 0$ . This contradicts that there is no such a disk as shown above. Thus  $E'$  is a regular neighborhood of  $D_1^0 \cup k_1 \cup D_1^1$ . Then, by the same reason as above, there is a  $\partial$ -parallel disk component of  $F_1$ , say  $E_1$ , which is parallel to a regular neighborhood of  $k_1$ . Since  $\partial E \cap \gamma_1 \neq \emptyset$ , there is an arc component of  $F \cap D_2$ , say  $\alpha$ , which connects a point in  $\partial E$  and a point in  $\gamma_2$ . Then by sliding  $\alpha$  (in fact by some ambient isotopy) as illustrated in Figure 13, we get a band  $b$  connecting  $E$  and  $E_1$ . Then the disk  $E \cup b \cup E_1$  is parallel to a disk in  $\partial V_1$  which satisfies the condition of Case (2)(i) in the proof of Lemma 2.4. This is a contradiction and completes the proof of Lemma 2.6.  $\square$

**Lemma 2.7** *Let  $A$  be a  $\partial$ -parallel annulus component of  $F_1$ , and  $A'$  the annulus in*

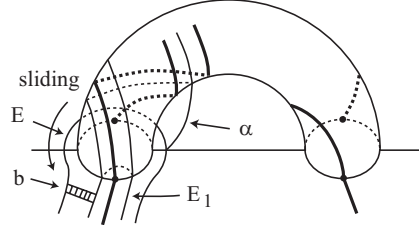


Figure 13

$\partial V_1$  to which  $A$  is parallel. Then we may assume that  $A$  is a small regular neighborhood of  $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$ .

**Proof.** Suppose there is an essential arc properly embedded in  $A'$ , say  $e$ , with  $e \cap (\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2) = \emptyset$ . Then by a boundary compression of  $A$  through  $e$ , we get a  $\partial$ -parallel disk. Then the disk satisfies the conclusion of Lemma 2.6, and by taking the number of annuli to be minimal, we may assume that there is no such an essential arc. Thus  $A' \cap (\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2)$  contains a central loop of  $A'$ . If one of  $\partial D_1^0$  and  $\partial D_1^1$  is a central loop of  $A'$ , say  $\partial D_1^0$ , then  $F_1 \cap D_1^0$  contains a loop component of  $\partial A'$ . This contradicts Lemma 2.5(1). Hence it is not  $\partial D_1^0$  or  $\partial D_1^1$ , and hence it is  $k_1 \cup k_2 \cup$  (a subarc of  $\partial D_1^0$ )  $\cup$  (a subarc of  $\partial D_1^1$ ). If  $\partial D_1^0 \not\subset A'$  or  $\partial D_1^1 \not\subset A'$ , then we can find a 2-gon in  $\partial A' \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$ . This contradicts Lemma 2.3, and shows that  $(\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2) \subset A'$ . This completes the proof of Lemma 2.7.  $\square$

### 3. Proof of Theorem 1.1

Since  $F_1$  is incompressible in  $V_1$ , each component of  $F_1$  is a  $\partial$ -parallel disk which is a small regular neighborhood of  $k_1$  as in Lemma 2.6, a  $\partial$ -parallel annulus which is a small regular neighborhood of  $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$  as in Lemma 2.7 or a meridian disk of  $V_1$ . Hence we have the following two subcases.

Case I:  $F_1 = \tilde{E} \cup \tilde{A}$ , where  $\tilde{E}$  consists of  $\partial$ -parallel disks and  $\tilde{A}$  consists of  $\partial$ -parallel annuli.

Case II:  $F_1 = \tilde{E} \cup \tilde{G}$ , where  $\tilde{E}$  consists of  $\partial$ -parallel disks and  $\tilde{G}$  consists of meridian disks.

Suppose we are in Case I. If  $\tilde{A} = \emptyset$ , then  $F_1 \cap \gamma_1 = \emptyset$  and  $F_2 = \emptyset$ . Hence  $\tilde{E} = \emptyset$  and  $F_1 = \emptyset$ . This means that  $F \cap K = \emptyset$ ,  $F = F_3$  and  $F \subset V_3$ . Then  $F$  is compressible, and this is a contradiction. Thus  $\tilde{A} \neq \emptyset$ , and then  $\tilde{E} \neq \emptyset$  because  $F_2 \neq \emptyset$  and  $n > 0$ . Therefore we can put  $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$  and  $\tilde{A} = A_1 \cup A_2 \cup \cdots \cup A_m$  for some  $m$ .

Since  $D_2 \cap (V_1 \cup V_2) = \partial D_2$ , we can take a product space  $D_2 \times I$  in  $V_3$  so that

$(D_2 \times I) \cap (V_1 \cup V_2) = \partial D_2 \times I$ . Put  $W_1 = (V_1 \cup V_2) \cup (D_2 \times I)$  and  $W_2 = cl(S^3 - W_1) = cl(V_3 - (D_2 \times I))$ . Then  $W_i$  ( $i = 1, 2$ ) is a solid torus and  $(W_1, W_2)$  is a genus one Heegaard splitting of  $S^3$ . Put  $H_1 = F \cap W_1$  and  $H_2 = F \cap W_2$ . Since each component of  $\partial H_1$  is a torus knot on  $\partial W_1$  in  $S^3$ , in the following, we detect the torus knot type.

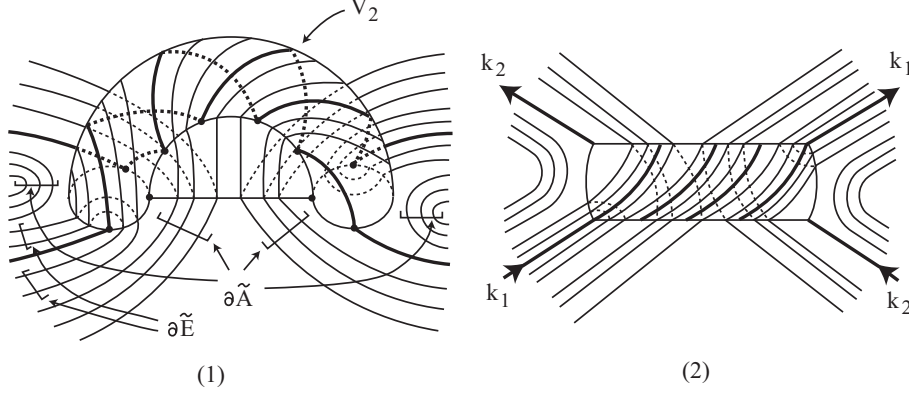


Figure 14

First, by a little observation, we have  $2m = n(2r - 1)$  with  $n = 2k$  for some integer  $k > 0$ . Figure 14(1) illustrates the local intersection of  $F_1 \cup F_2$  and  $D_2 \cup \partial(V_1 \cup V_2)$  in the case when  $n = 2$ , ( $k = 1$ ),  $m = 3$  and  $r = 2$ , and Figure 14(2) shows the bird's eye picture of  $\partial H_1 \cap \partial W_1$  from the higher position above  $V_2$ . Then, by this picture, we see that  $\partial H_1$  consists of two loops in this case. If  $n = 2k > 2$  (i.e.,  $k > 1$ ), then by considering the single loop in Figure 14(2) as a bunch of  $k$  loops, we can see that  $\partial H_1$  consists of  $n = 2k$  loops which correspond to  $n$  points  $\partial \tilde{E} \cap c_1$ . Since  $k_1 \cup c_1 \cup k_2 \cup c_2$  is a simple closed curve in the torus  $\partial W_1$ , we can put  $k_1 \cup c_1 \cup k_2 \cup c_2 = T(a, b)$  for some coprime integers  $(a, b)$ . Then, since  $T(a, b)$  is obtained from the given torus knot  $T(p, q)$  by a self-fusion along an arc which connects two points in parallel two strands of  $T(p, q)$  and intersects  $\gamma_1$  in a single point, we have  $pb - qa = \pm 2$  by [M4, Theorem 1.2], and  $(a, b)$  is uniquely determined by  $(p, q)$  because of  $0 \leq a < p$ . Next, since each component of  $\partial H_1$  is a torus knot in  $\partial W_1$ , we can denote it by  $T(s, t)$  for some coprime integers  $(s, t)$ . Then, since  $T(s, t)$  intersects  $T(a, b)$  in exactly two points with the same directions, we have  $at - bs = \pm 2$ , where one of the two points is in  $c_1$  and the other is in  $c_2$ .

Suppose  $t \neq 0$ . Then  $T(s, t)$  is not a longitude of  $W_1$  and is not a meridian of  $W_2$ . Thus each component of  $H_2$  is a  $\partial$ -parallel incompressible annulus in  $W_2$ . Let  $D_3$  be a meridian disk of  $W_2$  such that  $\partial D_3 \cap (V_2 \cup (D_2 \times I)) = \emptyset$  and  $\partial D_3$  is a longitude of  $V_1$ . Then by some ambient isotopy if necessary, we may assume that  $H_2 \cap D_3 (\neq \emptyset)$  consists of arcs properly embedded in  $D_3$  each of which is essential in  $H_2$ . Let  $\alpha$  be an outermost arc component of  $D_3 \cap H_2$  in  $D_3$ ,  $\Delta$  the outermost disk, and let  $A$  be

the component of  $H_2$  containing  $\alpha$ . Put  $\beta = cl(\partial\Delta - \alpha)$ . Since  $\beta$  connects different components of  $\partial H_1$ , and since  $\partial H_1$  consists of  $n$  loops corresponding to  $n$  points  $\partial\tilde{E} \cap c_1$ , by retaking  $\Delta$  in the solid torus in  $W_2$  cut off by  $A$  if necessary, we may assume that  $\beta \subset \partial V_1 - \partial(V_2 \cup (D_2 \times I))$  and  $\partial\beta \subset \partial\tilde{E}$ . If  $\beta \cap K = \emptyset$ , then by boundary compression along  $\Delta$ , we get a 2-gon in  $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$ , and have a contradiction by the same argument as the proof of Lemma 2.3 (4). If  $\beta \cap K \neq \emptyset$ , then, by taking an arc  $\alpha'$  in the component of  $\tilde{E}$  with  $\partial\alpha' = \partial\beta$ , the loop  $\alpha \cup \alpha'$  in  $F$  bounds a disk intersecting  $K$  in a single point. Then, by Lemma 2.1 and the argument in the proof of Lemma 2.4(1) Case (1), we get a contradiction. This shows that the case when  $t \neq 0$  does not occur.

Next, suppose  $t = 0$ . Then take the dual presentation of  $T(p, q)$ . Namely, take  $T(q, p)$  instead of  $T(p, q)$ . Then, since the exterior of  $T(q, p)$  is homeomorphic to the exterior of  $T(p, q)$ , we can perform the same arguments for the surface  $F$  as those in section 2, and we have the following two cases : (i)  $F_1 = \tilde{E} \cup \tilde{A}$  and (ii)  $F_1 = \tilde{E} \cup \tilde{G}$ . Suppose we are in case (i). Then by performing the same arguments as those in Case I, we get the torus knot  $T(b, a)$  from  $T(q, p)$  because  $(a, b)$  is uniquely determined by  $(p, q)$ . Then, as the next step, we get the torus knot  $T(t', s')$  from  $T(b, a)$  with  $bs' - at' = \pm 2$ . Since we are in the case when  $t = 0$ , we have  $s = \pm 1$  and  $b = \pm 2$  because of  $at - bs = \pm 2$ . If  $s' = 0$ , then  $t' = \pm 1$  and  $a = \pm 2$ . This contradicts  $\gcd(a, b) = 1$ . Hence  $s' \neq 0$  and we can get a contradiction similarly to the case when  $t \neq 0$ . If we are in case (ii), then this case is treated in the following Case II. Thus Case I does not occur except for the final case which is remained in Case II.

Suppose we are in Case II. If  $\tilde{G} = \emptyset$ , then  $F_1 \cap \gamma_1 = \emptyset$  and  $F_2 = \emptyset$ . Hence  $\tilde{E} = \emptyset$  and  $F_1 = \emptyset$ , a contradiction. Thus  $\tilde{G} \neq \emptyset$  and we can put  $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_e$  and  $\tilde{G} = G_1 \cup G_2 \cup \cdots \cup G_g$ , where  $n = e + \#(\tilde{G} \cap D_1^0)$ . Note that  $n = \#(D_1^0 \cap F)$ . We divide this case into the following two subcases : II(1)  $\tilde{G} \cap (D_1^0 \cup D_1^1) = \emptyset$  and II(2)  $\tilde{G} \cap (D_1^0 \cup D_1^1) \neq \emptyset$ .

Suppose we are in case II(1). Recall the notations in Case I, i.e.,  $W_1 = (V_1 \cup V_2) \cup (D_2 \times I)$ ,  $W_2 = cl(S^3 - W_1) = cl(V_3 - (D_2 \times I))$ ,  $H_1 = F \cap W_1$  and  $H_2 = F \cap W_2$ . In addition, we can put  $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$  because of  $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1) = 0$ . The schematic picture of  $\tilde{G}, \tilde{E}, D_1^0, D_1^1, k_1, k_2$  and  $\gamma_1$  is illustrated in Figure 15.

Since each component of  $\partial H_1$  is a torus knot on  $\partial W_1$  in  $S^3$ , we detect the torus knot type similarly to the proof of Case I. Suppose  $\gamma_1$  intersects  $G_1$  in  $m$  points. Then, since  $G_1, G_2, \cdots, G_g$  are all mutually parallel meridian disks, we have  $\#(\gamma_1 \cap \tilde{G}) = gm$  and  $n(2r - 1) = gm$ . Figure 16 illustrates the local intersection of  $F_1 \cup F_2$  and  $D_2 \cup \partial(V_1 \cup V_2)$  in the case when  $n = 2$  and  $r = 2$ , and Figure 17 shows the bird's eye picture of  $\partial H_1 \cap \partial W_1$  from the higher position above  $V_2$ .

Figure 17(1) is the case when  $n = 2$ ,  $r = 2$ ,  $g = 6$ ,  $m = 1$  and  $\partial H_1$  consists of two

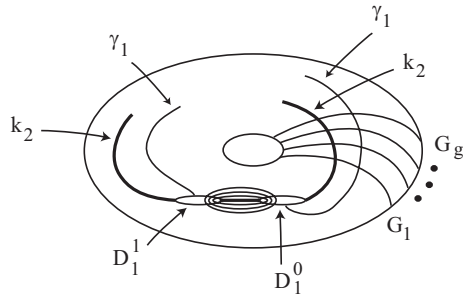


Figure 15

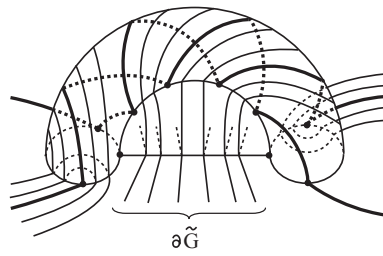


Figure 16

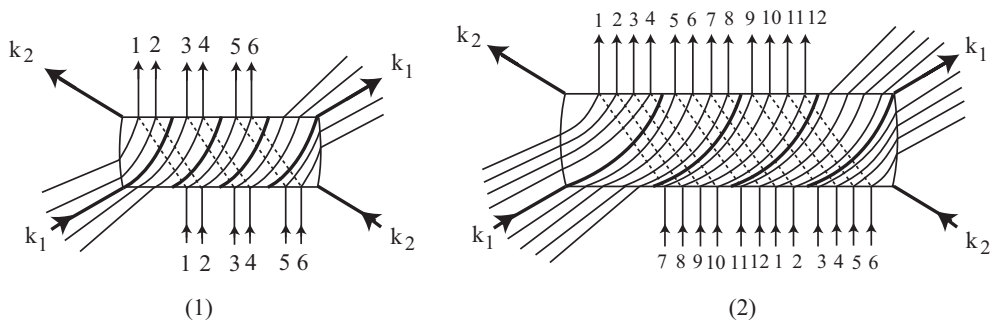


Figure 17

loops. Figure 17(2) is the case when  $n = 4$ ,  $r = 2$ ,  $g = 6$ ,  $m = 2$  and  $\partial H_1$  consists of two ( $= \frac{n}{2}$ ) loops. In general cases, by dividing the  $n$  arcs corresponding to the  $n$  points  $\#(\partial F_1 \cap c_1)$  into several bunchies, we see that  $\partial H_1$  consists of  $\ell$  loops for some divisor  $\ell$  of  $n$ .

Take a meridian-longitude loop pair in  $\partial V_1$  away from  $D_1^0 \cup \gamma_1 \cup D_1^1$  as illustrated in Figure 18. Count the algebraic intersection number of  $k_1$  and the meridian-longitude pair, and put the number  $(p_1, q_1)$ .

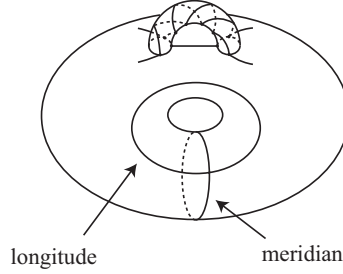


Figure 18

Then, since the algebraic intersection number of  $\tilde{G}$  and the meridian-longitude pair is  $(0, g)$ , the algebraic intersection number of  $\partial H_1$  and the meridian-longitude pair is  $(2np_1, 2nq_1 + g)$  because the  $2n$  arcs  $\partial \tilde{E} - (D_1^0 \cup D_1^1)$  are parallel to  $k_1$  (in the same direction as that of  $k_1$ ) as illustrated in Figure 17. Since  $\partial H_1$  consists of  $\ell$  loops, the intersection number of a single loop component and the meridian-longitude pair is  $(\frac{2np_1}{\ell}, \frac{1}{\ell}(2nq_1 + g))$ . Suppose  $\frac{1}{\ell}(2nq_1 + g) \neq 0$ . Then each component of  $\partial H_1$  is not a longitude of  $W_1$ , and is not a meridian of  $W_2$ . Thus each component of  $H_2$  is a  $\partial$ -parallel incompressible annulus in  $W_2$ . Then by the arguments similar to the proof of the case when  $t \neq 0$  in Case I, we can get a contradiction. Next suppose  $\frac{1}{\ell}(2nq_1 + g) = 0$ . Then  $g = -2nq_1$ , and by  $gm = n(2r - 1)$ , we have  $-2nq_1m = n(2r - 1)$ , i.e.,  $-2q_1m = 2r - 1$ . This is a contradiction, and shows that case II(1) does not occur.

Suppose we are in case II(2). Let  $D_3$  be a disk properly embedded in  $cl(S^3 - V_1)$  such that  $\partial D_3$  is a longitude of  $V_1$  with  $\partial D_3 \cap (V_2 \cup D_2) = \emptyset$ . Since  $\tilde{G} \neq \emptyset$  and each component of  $\tilde{G}$  is a meridian disk of  $V_1$ , we have  $\tilde{G} \cap D_3 \neq \emptyset$  and hence  $D_3 \cap F_3 \neq \emptyset$ , where  $F_3 = cl(F - (F_1 \cup F_2))$ . Then we may assume that each component of  $D_3 \cap F_3$  is an arc properly embedded in  $D_3$ . Let  $\alpha$  be an outermost arc component of  $D_3 \cap F_3$  in  $D_3$ ,  $\Delta$  the outermost disk in  $D_3$ , and put  $\beta = cl(\partial \Delta - \alpha)$ . Then  $\beta$  is an arc in  $\partial V_1$  with  $\beta \cap F_1 = \partial \beta$  and  $\beta \cap \gamma_1 = \emptyset$ . Moreover we may assume that there is no 2-gon in  $\beta \cup K$  which bounds a disk in  $\partial V_1$ .

**Lemma 3.1** *There is an outermost arc component  $\alpha$  of  $D_3 \cap F_3$  such that there is*

no 2-gon in  $\beta \cup \partial F_1$  which bounds a disk disjoint from  $K$  in  $\partial V_1$  and  $\beta$  satisfies one of the following : (1)  $\beta \cap K = \emptyset$ , (2)  $\partial\beta \subset \partial\tilde{E}$  or (3)  $\partial\beta \subset \partial\tilde{G}$ .

**Proof.** If there is a 2-gon in  $\beta \cup \partial F_1$  which bounds a disk disjoint from  $K$  in  $\partial V_1$ , then we can retake  $\alpha$  and  $\beta$  because of  $\partial D_3 \cap (V_2 \cup D_2) = \emptyset$ . Hence we can assume that there is no such a 2-gon. If  $\tilde{E} = \emptyset$ , then  $\partial\beta \subset \partial\tilde{G}$  and there is nothing to prove. Hence, we assume  $\tilde{E} \neq \emptyset$ . First we note that  $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1)$  because  $\#(F_1 \cap D_1^0) = \#(F_1 \cap D_1^1) = n$  and  $\#(\tilde{E} \cap D_1^0) = \#(\tilde{E} \cap D_1^1)$ . Then we may assume that  $E_1$  is the outermost component of  $\tilde{E}$  and that  $G_1$  is the component of  $\tilde{G}$  with  $G_1 \cap (D_1^0 \cup D_1^1) \neq \emptyset$  such that  $G_1$  is the closest component to  $E_1$  in  $D_1^0$  as in Figure 19.

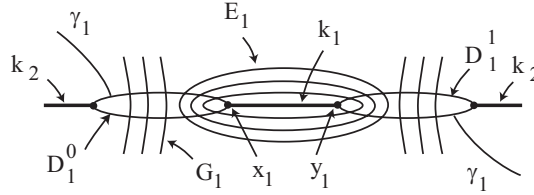


Figure 19

Let  $\alpha$  be an outermost arc component of  $D_3 \cap F_3$ , and  $\beta$  the corresponding arc in  $\partial D_3$ . Suppose  $\beta \cap K \neq \emptyset$  and  $\beta$  connects a component of  $\partial\tilde{E}$  and a component of  $\partial\tilde{G}$ . Then we have the following two subcases : case (a)  $\beta$  connects  $E_1$  and the component  $G_1$ , case (b)  $\beta$  connects  $E_1$  and a component  $G_2$  of  $\tilde{G}$  different from  $G_1$ .

Suppose we are in case (a). First suppose  $\beta$  meets  $G_1$  from the same side as the component  $D_1^0 \cap E_1$ . Then we have the situations as in Figure 20(1). If there are intersections of  $\beta$  and  $k_2$  as in Figure 20(2), then we can remove the intersection by sliding as in Figure 20(3), where such a sliding corresponds to an ambient isotopy to retake the disk  $D_3$ . Thus, the case when we need to consider is the situation as in Figure 21(1), i.e., the case when there is an intersection of  $D_1^1$  and  $\partial G_1$  which obstructs the sliding as in Figure 20(3). Then, since  $\gamma_1 \cap \beta = \emptyset$  and  $\gamma_1 \cap K = \emptyset$ , we have a 3-gon as illustrated in Figure 21(2). This is a contradiction by Lemma 2.5(2). Next suppose  $\beta$  meets  $G_1$  from the opposite side as the component  $D_1^0 \cap E_1$  as in Figure 22. Then  $\tilde{G} = G_1$  and this means  $F$  is a non-separating closed surface in  $S^3$ , a contradiction. Thus case (a) does not occur.

Next suppose we are in case (b). First we note that  $G_1$  and  $G_2$  are mutually parallel meridian disks and  $\beta$  is contained in the region between  $G_1$  and  $G_2$ . Then we have the following two subcases : case (b-1)  $G_2$  is the closest component to  $E_1$  in  $D_1^1$  as in Figure 23(1), case (b-2)  $G_1$  intersects  $D_1^1$  too and is the closest component to  $E_1$  in

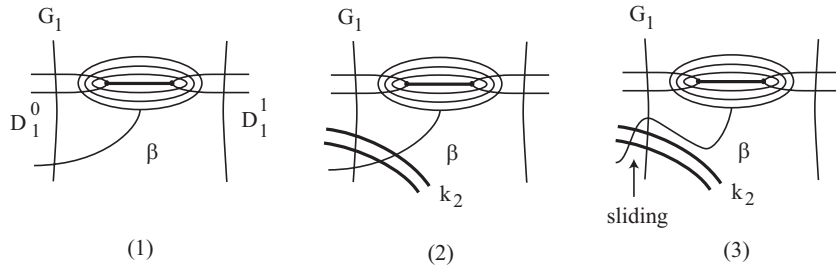


Figure 20

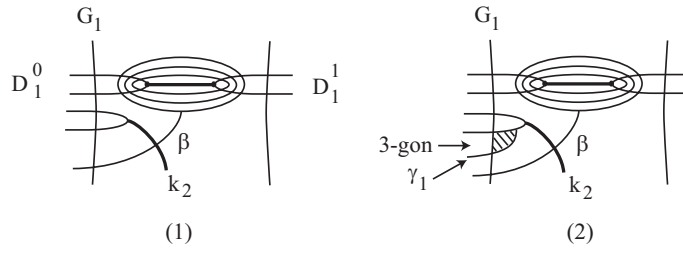


Figure 21

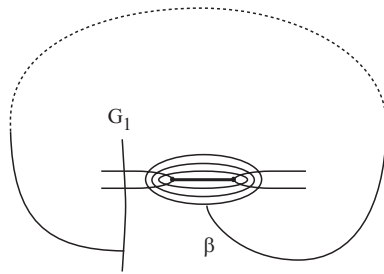


Figure 22

$D_1^1$  as in Figure 23(2).

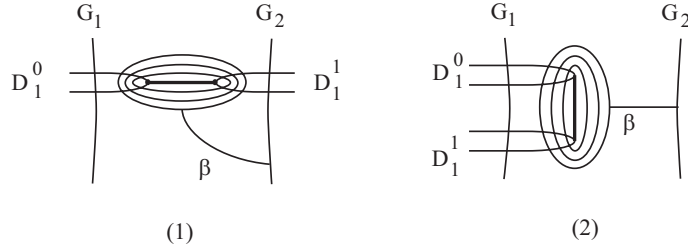


Figure 23

Suppose we are in case (b-1). Then, since  $\beta$  connects  $E_1$  and  $G_2$ , this case is the same one as in case (a), and we have a contradiction. Next suppose we are in case (b-2). Then,  $\beta$  can be slid to remove the intersection  $\beta \cap K$  as in Figure 20, or by noting that  $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1)$  we have the situation as in Figure 24, which is the case when there are intersections of  $G_2$  and  $D_1^0 \cup D_1^1$  which obstructs the sliding. Then, there is a 2-gon in  $\partial\tilde{G} \cup k_2$  as in Figure 24(1), which contradicts Lemma 2.3(2), or there is a 3-gon which consists of a subarc of  $\partial D_1^0 \cup \partial D_1^1$ , a subarc of  $\partial G_2$  and a subarc of  $k_2$  as in Figure 24(2), which contradicts Lemma 2.5(3). Hence  $k_2 \cap \tilde{G} = \emptyset$  as in Figure 24(3). Then, there is a 3-gon which consists of a subarc of  $\partial D_1^0 \cup \partial D_1^1$ , a subarc of  $\partial G_2$  and a subarc of  $\gamma_1$  as in Figure 24(3), which contradicts Lemma 2.5(2). Hence case (b) does not occur, and this completes the proof of Lemma 3.1.  $\square$

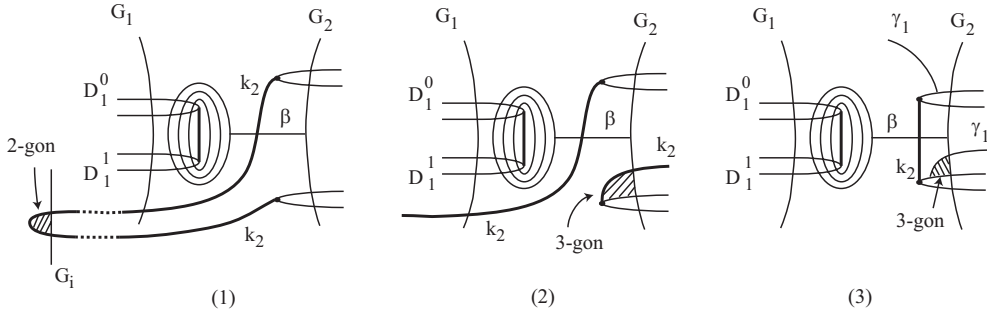


Figure 24

Now, let  $\alpha$  be an outermost arc component of  $F \cap D_3$  in  $D_3$ ,  $\Delta$  the outermost disk and  $\beta = cl(\partial\Delta - \alpha)$ . Then, by Lemma 3.1, we may assume that  $\beta$  satisfies one of the following three cases. Case II(2)-a :  $\beta \cap K = \emptyset$  and  $\beta$  connects a component of  $\partial\tilde{E}$  and a component of  $\partial\tilde{G}$ , Case II(2)-b :  $\partial\beta \subset \partial\tilde{E}$ , Case II(2)-c :  $\partial\beta \subset \partial\tilde{G}$ . Hereafter we assume that  $\#(F_1)$  is minimal among all essential surfaces  $F$  satisfying the condition of Case II.

Suppose we are in Case II(2)-a. In this case  $\beta \cap K = \emptyset$  and  $\beta$  connects a component of  $\tilde{E}$ , say  $E_1$ , and a component of  $\tilde{G}$ , say  $G_1$ . Let  $b$  be a band in  $V_1$  produced by a boundary compression along  $\Delta$ , and put  $G_0 = E_1 \cup b \cup G_1$ . Then  $G_0$  is a meridian disk properly embedded in  $V_1$  and  $G_0$  can be regarded as a component of  $\tilde{G}$ . This means that  $\#(\tilde{E})$  decreases by one and  $\#(\tilde{G})$  does not change, and contradicts the minimality of  $\#(F_1)$ . Thus Case II(2)-a does not occur.

Suppose we are in Case II(2)-b. In this case, since  $\beta \cap \tilde{G} = \emptyset$ ,  $\beta \cap \gamma_1 = \emptyset$  and  $\beta \cap (D_1^0 \cup D_1^1) = \emptyset$ , we have the three cases : ①  $\partial\beta$  connects different components of  $\partial\tilde{E}$ , ②  $\beta$  meets the innermost component of  $\partial\tilde{E}$ , ③  $\beta$  meets the outermost component of  $\partial\tilde{E}$  as illustrated in Figure 25.

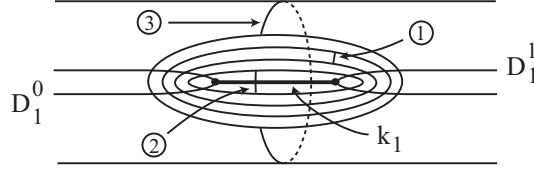


Figure 25

In case ①, by a boundary compression along  $\Delta$ , we get a 2-gon in  $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$  and we can decrease the number  $n$ , a contradiction. In case ②, we can get a disk as in Lemma 2.1, a contradiction. In case ③, let  $b$  be the band in  $V_1$  produced by a boundary compression along  $\Delta$ . Then, since  $b \cap \tilde{G} = \emptyset$ ,  $b \cup E_1$  is a compressible annulus in  $V_1$ , and let  $D$  be the compressing disk. Then, by the incompressibility of  $F - K$  in  $S^3 - K$ ,  $\partial D$  bounds a disk in  $F - K$ , say  $D'$ . Then by an ambient isotopy through the 3-ball bounded by  $D \cup D'$ , we can remove some components of  $F \cap V_2 = F_2$ . This contradicts the minimality of  $n = \#(F_2) = \#(F_1 \cap D_1^0)$ . Hence Case II(2)-b does not occur.

Suppose we are in Case II(2)-c. First suppose  $\beta \cap K = \emptyset$ , and let  $b$  be the band in  $V_1$  produced by a boundary compression along  $\Delta$ . Suppose  $b$  meets a single component of  $\tilde{G}$ , say  $G_1$ . If  $b$  meets  $G_1$  from both sides, then  $b \cup G_1$  is a Möbius band, a contradiction. If  $b$  meets  $G_1$  from one side, then there is a 2-gon in  $\beta \cup \partial G_1$  which bounds a disk disjoint from  $K$  in  $\partial V_1$ , a contradiction. Hence we may assume that  $b$  connects two components  $G_1$  and  $G_2$ . Then, since  $G_1$  and  $G_2$  are mutually parallel meridian disks of  $V_1$ , we have the following four cases related to the neighborhood of  $b$  : (1)  $b$  is parallel to  $k_2$ , (2)  $b$  is parallel to  $D_1^0$  or to  $D_1^1$ , (3)  $b$  is parallel to  $D_1^0 \cup k_2$  or to  $D_1^1 \cup k_2$ , (4)  $b$  is parallel to  $D_1^0 \cup k_1 \cup D_1^1$  as illustrated in Figure 26.

Put  $E_0 = G_1 \cup b \cup G_2$ . Then  $E_0$  is a  $\partial$ -parallel disk. In cases (1), (2), we have a contradiction by Lemma 2.3. In case (3), we have a contradiction by Lemma 2.4 (Figure 6). In case (4), if  $(\partial G_1 \cup \partial G_2)$  intersects  $k_2$  as in the cases (1) or (3), or

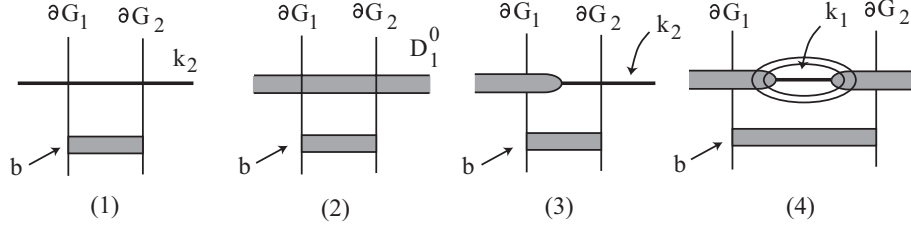


Figure 26

intersects  $D_1^0 \cup D_1^1$  as in case (2), then we have a contradiction similarly. Hence  $(\partial G_1 \cup \partial G_2) \cap k_2 = \emptyset$  and  $(\partial G_1 \cup \partial G_2) \cap (D_1^0 \cup D_1^1)$  consists of exactly two arcs as in Figure 26(4). Then  $E_0$  can be regarded as a component of  $\tilde{E}$ , and this means that  $\#(\tilde{E})$  increase by one and  $\#(\tilde{G})$  decrease by two. This contradicts the minimality of  $\#(F_1)$ , and case (4) does not occur.

Next suppose  $\beta \cap K \neq \emptyset$ . If we can remove the intersection by some sliding as in Figure 20, then this case can be regarded as the case when  $\beta \cap K = \emptyset$ . Hence we assume that  $\beta \cap K$  can not be removed by sliding. Put  $X = \{x_1, x_2, y_1, y_2\}$ .

First suppose  $\beta$  meets a single component of  $\tilde{G}$ , say  $G_1$ . Then the disk bounded by the 2-gon  $\beta \cup$  (a subarc of  $\partial G_1$ ) contains some points of  $X$ . If it contains three or four points of  $X$ , then, by considering the intersections of  $D_1^0 \cup D_1^1$  and  $\partial \tilde{G}$ , we can find a 2-gon in  $\partial \tilde{G} \cup ((\partial D_1^0 \cup \partial D_1^1) - X)$ , and this contradicts Lemma 2.3(4). Hence it contains one or two point(s) of  $X$  as in Figure 27. If it contains one point then we have a contradiction by Lemma 2.4 (Figure 6). Suppose it contains two points. If one of the two points is  $x_2$  or  $y_2$ , then we can find a 3-gon as in Lemma 2.5(2) (c.f. Figure 21(2)), a contradiction. Thus the two points are  $x_1$  and  $y_1$ . Such a case can occur in the case when  $\tilde{E} = \emptyset$ . Then, since  $k_1$  connects  $x_1$  and  $y_1$ , we can find a 2-gon in  $k_1 \cup \tilde{G}$  which bounds a disk in  $\partial V_1$  (c.f. Figure 24(1)). This contradicts Lemma 2.3(2).

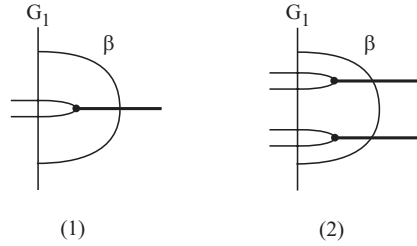


Figure 27

Next suppose  $\beta$  connects two different components of  $\tilde{G}$ , say  $G_1$  and  $G_2$ , and let  $R$  be the region (an annulus) in  $\partial V_1$  between  $G_1$  and  $G_2$  containing  $\beta$ . If  $R$  contains at

most one point of  $X$ , then we can remove intersections of  $\beta$  and  $K$  by sliding. Hence  $R \cap X$  consists of two, three or four points.

Suppose  $\#(R \cap X) = 2$ . Then we have the three situations illustrated in Figure 28, which are the case when  $\beta \cap K$  cannot be removed by sliding. Suppose we are in the situation of Figure 28(1). If at least one of the two points of  $X$  in  $R$  is  $x_2$  or  $y_2$ , then we can find a 3-gon as in Lemma 2.5(2), a contradiction. Hence the two points are  $x_1$  and  $y_1$ . Then, since there are at least two outermost components of  $F_3 \cap D_3$ , take another outermost component, say  $\alpha'$ , and let  $\beta'$  be the corresponding arc in  $\partial D_3$ . If  $\beta'$  is contained in  $R$ , then, since  $\beta$  and  $\beta'$  are mutually parallel arcs in  $R$ , and since  $\partial D_3$  is a longitude of  $V_1$ , we can find a 2-gon in  $\partial D_3 \cup \partial \tilde{G}$  which bounds a disk in  $\partial V_1$ . Then the disk contains  $x_2$  or  $y_2$  and we can find a 3-gon as in Lemma 2.5(2), a contradiction. If  $\beta'$  is contained in the region different from  $R$ , say  $R'$ , then we can remove the intersection of  $\beta'$  and  $K$ , or the same situations as in Figure 28 happen. In the former case, we have a contradiction similarly to the case when  $\beta \cap K = \emptyset$ . In the latter case, we can find a 3-gon as in Lemma 2.5(2) or (3) because  $R' \cap X = \{x_2, y_2\}$ , and this is a contradiction.

Finally, consider the case when  $\#(R \cap X) = 3$  or 4. Then by the arguments similar to the above, we can find a 3-gon as in Lemma 2.5(2) or (3), a contradiction. Thus Case II(2)-c does not occur, and hence Case II does not occur. After all, either Case I nor Case II occurs, and we complete the proof of Theorem 1.1.  $\square$

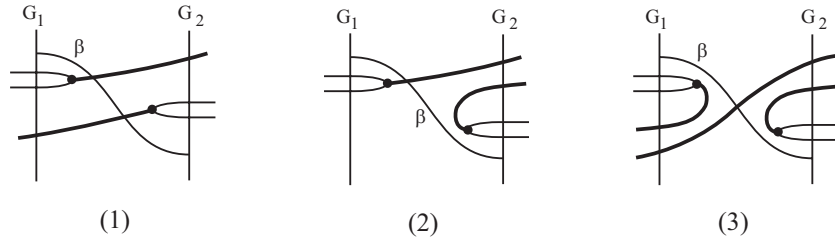


Figure 28

## References

- [CGLS] M. Culler, C. McA. Gordon, J. Lueke and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. **115** (1987) 237-300
- [Ho] P. Hoiden, *On 1-bridge genus of small knots*, Topology Appl. **106** (2000) 321-335
- [M1] K. Morimoto, *On the super additivity of tunnel number of knots*, Math. Ann. **317** (2000) 489-508
- [M2] K. Morimoto, *Characterization of composite knots with 1-bridge genus two*,

- J. K. T. R. **10** (2001) 823-840
- [M3] K. Morimoto, *Tunnel number, 1-bridge genus and h-genus of knots*,  
Topology Appl. **146** (2005) 149-158
- [M4] K. Morimoto, *A note on self-fusion of torus knots*, Memoirs of Konan University  
Science and Engineering Series, **53** (2006) 1-8
- [MSY] K. Morimoto, M. Sakuma and Y. Yokota, *Examples of tunnel number one  
knots which have the property "1 + 1 = 3"*, Math. Proc. Camb. Phil. Soc.  
**119** (1996) 113-118
- [MY] K. Morimoto and Y. Yamada, *A note on essential tori in the exteriors of torus  
knots with twists*, Preprint
- [No] F. Norwood, *Every two generator knot is prime*,  
Proc. A. M. S. **86** (1982) 143-147
- [Ro] D. Rolfsen, *Knots and Links*, AMS Chelsea Publishing (1976, 2003)
- [Sc] M. Scharlemann, *Tunnel number one knots satisfy the Poenaru conjecture*,  
Topology Appl. **18** (1984) 235-258